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Published in:
Journal of Physics: Conference Series

DOI:
10.1088/1742-6596/1194/1/012055

Publication date:
2019

Document version
Publisher's PDF, also known as Version of record

Journal of Physics: Conference Series

PAPER • OPEN ACCESS

Special classes of homomorphisms between generalized Verma modules for $\mathcal{U}_q(su(n,n))$

Special classes of homomorphisms between generalized Verma modules for $\mathcal{U}_q(su(n,n))$

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Abstract. We study homomorphisms between quantized generalized Verma modules $M(V_\Lambda) \overset{\phi_{\Lambda,\Lambda_1}}{\to} M(V_{\Lambda_1})$ for $\mathcal{U}_q(su(n,n))$. There is a natural notion of degree for such maps, and if the map is of degree k, we write $\phi^k_{\Lambda,\Lambda_1}$. We examine when one can have a series of such homomorphisms $\phi^1_{\Lambda_{n-1},\Lambda_n} \circ \phi^1_{\Lambda_{n-2},\Lambda_{n-1}} \circ \cdots \circ \phi^1_{\Lambda,\Lambda_1} = \mathrm{Det}_q$, where Det_q denotes the map $M(V_\Lambda) \ni p \to \det_q \cdot p \in M(V_{\Lambda_n})$. If, classically, $su(n,n)^{\mathbb{C}} = \mathfrak{p}^- \oplus (su(n) \oplus su(n) \oplus \mathbb{C}) \oplus \mathfrak{p}^+$, then $\Lambda = (\Lambda_L, \Lambda_R, \lambda)$ and $\Lambda_n = (\Lambda_L, \Lambda_R, \lambda + 2)$. The answer is then that Λ must be one-sided in the sense that either $\Lambda_L = 0$ or $\Lambda_R = 0$ (non-exclusively). There are further demands on λ if we insist on $\mathcal{U}_q(\mathfrak{g}^{\mathbb{C}})$ homomorphisms. However, it is also interesting to loosen this to considering only $\mathcal{U}_q^-(\mathfrak{g}^{\mathbb{C}})$ homomorphisms, in which case the conditions on λ disappear. By duality, there result have implications on covariant quantized differential operators. We finish by giving an explicit, though sketched, determination of the full set of $\mathcal{U}_q(\mathfrak{g}^{\mathbb{C}})$ homomorphisms $\phi^1_{\Lambda,\Lambda_1}$.

Dedicated to I.E. Segal (1918-1998) in commemoration of the centenary of his birth.

1. Introduction

Generalized and quantized Verma modules have physically attractive properties similar to the Fock space. There is a "vacuum vector", here called a highest weight vector, which is annihilated by the "upper diagonal" operators, is an eigenvector for the "diagonal operators", and which generate the whole space when acted upon by the algebra of "lower diagonal operators". Since it may happen that there is a second vacuum vector, it is of interest to determine cases in which this may happen. This is further interesting because by duality, such cases correspond to quantized covariant differential operators such as the Maxwell equations. We give here a complete proof of the one-sidedness and we give a sketch of the case of an arbitrary first order. Further details as well as the dual picture will appear in a forthcoming article. For the "classical" analogue, see e.g. [2]. On a personal note: The explicitness presented here is in line with how mathematical physics was taught to me by Segal, my Ph.D. advisor.

2. Set-up

$$\begin{array}{lclclclcl} \mathfrak{g}^{\mathbb{C}} & = & su(n,n)^{\mathbb{C}} & = & \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p} & = & \mathfrak{p}^- \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+ & = & \mathfrak{p}^- \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+, \\ \mathfrak{k}^{\mathbb{C}} & = & su(n)^{\mathbb{C}} \oplus \mathbb{C} \oplus su(n)^{\mathbb{C}} & = & \mathfrak{k}^{\mathbb{C}}_L \oplus \zeta \oplus \mathfrak{k}^{\mathbb{C}}_R \,. & & & & \end{array} \tag{1}$$

$$\zeta \text{ is the center}, \quad \mathfrak{p}^\pm \text{ are abelian } \mathcal{U}(\mathfrak{k}^{\mathbb{C}}) \text{ modules}, \quad \text{and} \quad \mathcal{U}(\mathfrak{g}^{\mathbb{C}}) = \mathcal{P}(\mathfrak{p}^-) \cdot \mathcal{U}(\mathfrak{k}^{\mathbb{C}}) \cdot \mathcal{P}(\mathfrak{p}^+) \,. \tag{2}$$

We let the simple roots be denoted $\Pi = \{\mu_1, \ldots, \mu_{n-1}\} \cup \{\beta\} \cup \{\mu_1, \ldots, \mu_{n-1}\}$, where β is the unique non-compact roots and where the decomposition of simple roots corresponds to the decomposition of $\mathfrak{k}^{\mathbb{C}}$ above. In the quantum group $\mathcal{U}_q(\mathfrak{g}^{\mathbb{C}})$, we denote the generators by $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ for $\alpha \in \Pi$. There are also decompositions

$$\mathcal{U}_q(\mathfrak{g}^{\mathbb{C}}) = \mathcal{A}_q^- \cdot \mathcal{U}_q(\mathfrak{k}^{\mathbb{C}}) \cdot \mathcal{A}_q{}^+, \tag{3}$$
$$\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}}) = \mathcal{U}_q(\mathfrak{k}_L^{\mathbb{C}}) \cdot \mathbb{C}[K_\beta^{\pm 1}] \cdot \mathcal{U}_q(\mathfrak{k}_R^{\mathbb{C}}). \tag{4}$$

Here, $\mathcal{A}_q^\pm$ are quadratic algebras which are furthermore $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$. Specifically,

$$\mathcal{A}_q^- = \mathbb{C}[W_{i,j} \mid i,j = 1, \ldots, n\}, \tag{5}$$
$$\mathcal{A}_q^+ = \mathbb{C}[Z_{i,j} \mid i,j = 1, \ldots, n\}, \tag{6}$$

with relations

$$Z_{ij}Z_{ik} = q^{-1}Z_{ik}Z_{ij} \text{ if } j < k; \tag{7}$$
$$Z_{ij}Z_{kj} = q^{-1}Z_{kj}Z_{ij} \text{ if } i < k; \tag{8}$$
$$Z_{ij}Z_{st} = Z_{st}Z_{ij} \text{ if } i < s \text{ and } t < j; \tag{9}$$
$$Z_{ij}Z_{st} = Z_{st}Z_{ij} - (q - q^{-1})Z_{it}Z_{sj} = \text{ if } i < s \text{ and } j < t. \tag{10}$$

The algebra $\mathcal{A}_q^-$ have the same relations, but the algebras $\mathcal{A}_q^\pm$ are different as $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ modules. The elements Z_{ij} and W_{ij} are constructed by means of the Lusztig operators. References [4] and [3] are general references of much of this. Using the Serre relations one gets, setting $\mu_0 = Id$,

Lemma 2.1.

$$Z_{i,j} = T_{\nu_{j-1}}T_{\nu_{j-2}} \ldots T_{\nu_0} \cdot T_{\mu_{i-1}}T_{\mu_{i-2}} \ldots T_{\mu_0}(E_\beta), \tag{11}$$
$$W_{i,j} = T_{\nu_{j-1}}T_{\nu_{j-2}} \ldots T_{\nu_0} \cdot T_{\mu_{i-1}}T_{\mu_{i-2}} \ldots T_{\mu_0}(F_\beta). \tag{12}$$

For later use, we give the relations in the full algebra:

$$E_{\mu_k}W_{i,j} = W_{i,j}E_{\mu_k} \text{ if } k \neq i-1, \tag{13}$$
$$E_{\mu_k}W_{i,j}^a = (-q)[a]W_{i-1,j}W_{i,j}^{a-1}K_{\mu_k} + W_{i,j}^aE_{\mu_k} \text{ if } k = i-1, \tag{14}$$
$$F_{\mu_k}W_{i,j} = W_{i,j}F_{\mu_k} \text{ if } k \neq i, i-1, \tag{15}$$
$$F_{\mu_k}W_{i,j}^a = -q^{-1}[a]W_{i,j}^{a-1}W_{i+1,j} + q^{-a}W_{i,j}^aF_{\mu_k} \text{ if } k = i, \tag{16}$$
$$F_{\mu_k}W_{i,j} = qW_{i,j}F_{\mu_k} \text{ if } k = i-1, \tag{17}$$
$$F_{\mu_k}Z_{i,j} = Z_{i,j}F_{\mu_k} \text{ if } k \neq i-1, \tag{18}$$
$$F_{\mu_k}Z_{i,j}^a = [a]Z_{i-1,j}Z_{i,j}^{a-1}K_{\mu_k}^{-1} + W_{i,j}^aE_{\mu_k} \text{ if } k = i-1, \tag{19}$$
$$E_{\mu_k}Z_{i,j} = Z_{i,j}E_{\mu_k} \text{ if } k \neq i, i-1, \tag{20}$$
$$E_{\mu_k}Z_{i,j}^a = [a]Z_{i,j}^{a-1}Z_{i+1,j} + q^{-a}Z_{i,j}^aE_{\mu_k} \text{ if } k = i, \tag{21}$$
$$E_{\mu_k}Z_{i,j} = qZ_{i,j}E_{\mu_k} \text{ if } k = i-1. \tag{22}$$

There are similar formulas for the commutators involving E_{ν_k} and F_{ν_k}. If e.g. S denotes the obvious automorphism defined on generators by $W_{ij} \to W_{j,i}$, and similarly, $Z_{ij} \to Z_{j,i}$ then $E_{\nu_k} = SE_{\mu_k}S$ and $F_{\nu_k} = SF_{\mu_k}S$.

3. Finite dimensional $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ modules

A non-zero vector v_Λ of a finite dimensional module V_Λ of $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ is a highest weight vector of highest weight Λ, and V_Λ is a highest weight module of highest weight λ, if

$$\begin{array}{lll}\forall i = 1, \ldots, n-1 : K_{\mu_i}^{\pm 1} = q^{\pm \lambda_i^\mu} v_\Lambda, & K_{\nu_i}^{\pm 1} = q^{\pm \lambda_i^\nu} v_\Lambda, & \text{and } K_\beta^{\pm 1} = q^{\pm \lambda} v_\Lambda. \\ \text{Finally, } \mathcal{U}_q^+(\mathfrak{k}^{\mathbb{C}}) v_\Lambda = 0, & \text{and } \mathcal{U}_q^-(\mathfrak{k}^{\mathbb{C}}) v_\Lambda = V. & \end{array} \tag{23}$$

We set $\Lambda = ((\lambda_1^\mu, \ldots, \lambda_{n-1}^\mu), (\lambda_1^\nu, \ldots, \lambda_{n-1}^\nu); \lambda) = (\Lambda_L, \Lambda_R, \lambda)$.

As a vector space, $V_\Lambda = V_{\Lambda_L} \otimes V_{\Lambda_R}$ where V_{Λ_L} and V_{Λ_R} are highest weight representations of $\mathcal{U}_q(\mathfrak{k}_L^{\mathbb{C}})$ and $\mathcal{U}_q(\mathfrak{k}_R^{\mathbb{C}})$, respectively, of highest weights $\Lambda_L = (\lambda_1^\mu, \ldots, \lambda_{n-1}^\mu)$ and $\Lambda_R = (\lambda_1^\nu, \ldots, \lambda_{n-1}^\nu)$, respectively. The highest weight vector can then be written as $v_\Lambda = v_{\Lambda_L} \otimes v_{\Lambda_R}$ with the stipulation that $K_\beta^{\pm 1} v_{\Lambda_L} \otimes v_{\Lambda_R} = q^{\pm \lambda} v_{\Lambda_L} \otimes v_{\Lambda_R}$.

4. Generalized quantized Verma modules and their homomorphisms

Consider a finite dimensional module $V_\Lambda = V_{\Lambda_L, \Lambda_R, \lambda}$ over $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ with highest weight is defined by $\Lambda = (\Lambda_L, \Lambda_R, \lambda)$ where $\Lambda_L = (\lambda_1^\mu, \lambda_2^\mu, \ldots, \lambda_{n-1}^\mu, 0)$, $\Lambda_R = (\lambda_1^\nu, \lambda_2^\nu, \ldots, \lambda_{n-1}^\nu, 0)$, and $\lambda \in \mathbb{C}$.

We extend such a module to a $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})\mathcal{A}_q^+$ module, by the same name, by letting $\mathcal{A}_q^+$ act trivially.

Definition 4.1. *The quantized generalized Verma module $M(V_\Lambda)$ is given by*

$$M(V_\Lambda) = \mathcal{U}_q(\mathfrak{g}^{\mathbb{C}}) \bigotimes_{\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})\mathcal{A}_q^+} V_\Lambda \tag{24}$$

with the natural action from the left.

As a vector space,

$$M(V_\Lambda) = \mathcal{A}_q^- \otimes V_\Lambda. \tag{25}$$

We are interested in structure preserving homomorphisms between quantized generalized Verma modules. We call such maps intertwiners, covariants, or equivariants, indiscriminately. Dually, they will be quantized covariant differential operators. In abstract notation, the structure under investigation is

$$Hom_{\mathcal{U}_q(\mathfrak{g}^{\mathbb{C}})}(M(V_\Lambda), M(V_{\Lambda_1})). \tag{26}$$

However, for the time being we will consider

$$Hom_{\mathcal{A}_q^- \mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})}(M(V_\Lambda), M(V_{\Lambda_1})). \tag{27}$$

An element ϕ_{Λ,Λ_1} in the latter space is completely determined by the $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ equivariant map, denoted by the same symbol:

$$V_\Lambda \overset{\phi_{\Lambda,\Lambda_1}}{\to} \mathcal{A}_q^- \otimes V_{\Lambda_1} \text{ leads to } \mathcal{A}_q^- \otimes V_\Lambda \overset{\phi_{\Lambda,\Lambda_1}}{\to} \mathcal{A}_q^- \otimes V_{\Lambda_1}. \tag{28}$$

Specifically, ϕ_{Λ,Λ_1} does not depend on λ and is completely given by the condition that the image of the highest weight vector $\phi_{\Lambda,\Lambda_1}(v_\Lambda)$ is a highest weight vector for $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$. For the map ϕ_{Λ,Λ_1} to belong to the former space (27) it is necessary, and sufficient that, additionally, (Z_β acting in $M(V_{\Lambda_1})$)

$$Z_\beta\left(\phi_{\Lambda,\Lambda_1}(v_\Lambda)\right) = 0. \tag{29}$$

This equation depends heavily on λ. It is clear that such maps, whether of the first or second kind, can be combined:

$$\phi_{\Lambda_1,\Lambda_2} \circ \phi_{\Lambda,\Lambda_1} = \phi_{\Lambda,\Lambda_2} \tag{30}$$

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though it may happen that the composite is zero.

We use the terminology of degree of elements of $\mathcal{A}_q^-$ in the obvious way, and we let, for $k = 1, \dots,$ $\mathcal{A}_q^-(k)$ denote the $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ module spanned by homogeneous elements of degree k. If the elements p_{ij} all belong to $\mathcal{A}_q^-(k)$, we write $\phi^k_{\Lambda,\Lambda_1}$.

General Problem: When is it possible to have $\phi^1_{\Lambda_{n-1},\Lambda_n} \circ \phi^1_{\Lambda_{n-2},\Lambda_{n-1}} \circ \cdots \circ \phi^1_{\Lambda,\Lambda_1} = \mathrm{Det}_q$? In this case, if $\Lambda = (\lambda_L, \Lambda_R, \lambda)$, then $\Lambda_n = (\Lambda_L, \Lambda_R, \lambda + 2)$.

5. Laplace expansion

If $m = n$, one may define the quantum determinant det_q in $\mathcal{A}_q-$ as follows:

$$\begin{aligned} \det_q(n) = \det_q &= \Sigma_{\sigma\in S_n}(-q^{-1})^{\ell(\sigma)} W_{1,\sigma(1)}W_{2,\sigma(2)}\cdots W_{n,\sigma(n)} && (31)\\ &= \Sigma_{\delta\in S_n}(-q^{-1})^{\ell(\delta)} W_{\delta(1),1}W_{\delta(2),2}\cdots W_{\delta(n),n}. && (32)\end{aligned}$$

If $m = n$ and $I = \{i_1 < 1_2 < \cdots < i_{n-1}\} = \{1, 2, \cdots, n\} \setminus \{i\}$, $J = \{j_1 < j_2 < \cdots < j_{n-1}\} = \{1, 2, \cdots, n\} \setminus \{j\}$, we set

$$\begin{aligned} A(i,j) &= \Sigma_{\sigma\in S_{n-1}}(-q^{-1})^{\ell(\sigma)} W_{i_1,j_{\sigma(1)}}W_{i_2,j_{\sigma(2)}}\cdots W_{i_{n-1},j_{\sigma(n-1)}} && (33)\\ &= \Sigma_{\tau\in S_{n-1}}(-q^{-1})^{\ell(\tau)} W_{i_{\tau(1)},j_1}W_{i_{\tau(2)},j_2}\cdots W_{i_{\tau(n-1)},j_{n-1}}. && (34)\end{aligned}$$

These elements are quantum $(n-1) \times (n-1)$ minors. The following was proved by Parshall and Wang [6]:

Proposition 5.1. $\det_q$ *is central. Furthermore, let* $i, k \le n$ *be fixed integers. Then*

$$\begin{aligned} \delta_{i,k}\det_q &= \sum_{j=1}^{n}(-q^{-1})^{j-k}W_{i,j}A(k,j) = \sum_j(-q^{-1})^{i-j}A(i,j)W_{k,j} && (35)\\ &= \sum_j(-q^{-1})^{j-k}W_{j,i}A(j,k) = \sum_j(-q^{-1})^{i-j}A(j,i)W_{j,k}. && (36)\end{aligned}$$

6. 1. order

Any finite dimensional highest weight representation of $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ of the form $\Lambda = (\Lambda_L, \Lambda_R, \lambda)$ in which either $\Lambda_L = 0$ or $\Lambda_R = 0$ will be called one-sided. We will now give an explicit form for a highest weight vector v_1 of an irreducible sub-representation of $\mathcal{A}^-(1) \otimes V_{\Lambda=(\Lambda_L,0,\lambda)}$. Specifically, consider

$$v_1 = W_{N+1,1}v_0 + W_{N,1}u_N v_0 + W_{N-1,1}u_{N-1,1}v_0 + W_{N-2,1}u_{N-2}v_0 + \cdots + W_{1,1}u_1v_0, \qquad (37)$$

where $\forall i = 1, \dots, N : u_i \in \mathcal{U}_q^{-\mu_N-\mu_{N-1}-\cdots-\mu_i}(\mathfrak{k}^{\mathbb{C}})$.

Because of this, we first want to consider a basis of $\mathcal{U}_q^{-\mu_N+\cdots-\mu_\ell}(\mathfrak{k}_L^{\mathbb{C}})$.

Set $\mathcal{E}_{\ell,N} = \{\ell, \ell+1, \cdots, N\} \subseteq \{1, 2, \dots, n-1\}$. Any sequence $I_{\ell,N} = (i_\ell, i_{\ell+1}, \cdots, i_N)$ made up of pairwise different elements of $\mathcal{E}^\mu_{\ell,N}$ defines a non-zero element

$$F_{\mu_{i_\ell}}F_{\mu_{i_{\ell+1}}}\cdots F_{\mu_{i_N}} = F^\mu(I_{\ell,N}) \in \mathcal{U}_q^{-\mu_\ell+\cdots-\mu_N}(\mathfrak{k}_L^{\mathbb{C}}). \qquad (38)$$

We will call such a sequence **allowed**. We reserve the name $E_{\ell,N}$ for the special sequence $(\ell, \ell+1, \cdots, N)$.

We will say that a transposition $(i_\ell, i_{\ell+1}, \ldots, i_k, i_{k+1}, \ldots, i_N) \to (i_\ell, i_{\ell+1}, \ldots, i_{k+1}, i_k, \ldots, i_N)$ is legal if $\mid i_{k+1} - i_k \mid > 1$.

Recall that $F_{\mu_i} F_{\mu_j} = F_{\mu_j} F_{\mu_i}$ if $|i-j| > 1$. We will say that two allowed sequences $I^{(1)}$ and $I^{(2)}$ are equivalent if one can be obtained from the other by a series of legal transpositions. It is clear that any allowed sequence I can be brought, uniquely, and by legal transpositions, into the form $J_1 J_2 \cdots J_r$ which is the concatenation of sequences J_t that are either descending or ascending, and such that the following are satisfied: Firstly, the elements of J_s are smaller than the elements of J_t if $s < t$, and $\cup_s J_s = \{\ell, \ell+1, \ldots, N\}$. Secondly, two neighboring sequences cannot both be ascending (maximality), and thirdly, singletons are ascending.

We denote by $\mathcal{J}_{\ell,N}$ the set of such sequences. The following is then obvious:

Proposition 6.1.

$$\{F^\mu(I_{\ell,N}) \mid I_{\ell,N} \in \mathcal{J}_{\ell,N}\} \tag{39}$$

is a basis of $\mathcal{U}_q^{-\mu_\ell + \cdots - \mu_N}(\mathfrak{k}_L^{\mathbb{C}})$.

We furthermore have from e.g. [1, lemma 6.27]:

Proposition 6.2. *Let* $V = V(\Lambda_L)$ *be a finite dimensional highest weight representation of* $\mathcal{U}_q(\mathfrak{k}_L^{\mathbb{C}})$ *with* $\Lambda_L = (\lambda_1^\mu, \lambda_2^\mu, \cdots, \lambda_{n-1}^\mu)$ *satisfying:* $\lambda_\ell^\mu > 0, \lambda_{\ell+1}^\mu > 0, \ldots, \lambda_N^\mu > 0$. *Let* v_0 *denote a highest weight vector (unique up to a non zero constant). Then*

$$\{F^\mu(I_{\ell,N}) v_0 \mid I_{\ell,N} \in \mathcal{J}_{\ell,N}\} \tag{40}$$

is a basis of $V^{\Lambda_L - \mu_\ell + \cdots - \mu_N}$.

If $I_{\ell,N} = J_1 J_2 \cdots J_s \in \mathcal{J}_{\ell,N}$ as above, we attach to it a sequence $C^\mu(I_{\ell,N}) = (c_{i_\ell}, c_{i_{ell+1}}, \cdots c_{i_N})$ where $c_k = a_k$ if either i_k belongs to an ascending sub-sequence J_x of $I_{\ell,N}$ or if i_k is the biggest element in a descending sub-sequence J_y of $I_{\ell,N}$. Here, $x, y \in \{1, 2, \ldots, s\}$. In the remaining cases, $c_{i_k} = b_{i_k}$. We furthermore set $f^\mu(C^\mu(I_{\ell,N})) = \prod_{t=\ell}^N c_{i_t}$.

We can then state, maintaining the assumptions from Lemma 6.2:

Proposition 6.3. *If the vector* v_1 *in (37) is a highest weight vector in* $\mathcal{A}_1^- \otimes V(\Lambda_L)$ *then*

$$\forall \ell = 1, \ldots, N : u_\ell = \sum_{I_{\ell,N} \in \mathcal{J}_{\ell,N}} f^\mu(C^\mu(I_{\ell,N})) F^\mu(I_{\ell,N}) v_0. \tag{41}$$

Later, we shall find it convenient to set $\mathcal{J}_{N+1,N} = \emptyset$ and $f^\mu(C^\mu(\emptyset)) = 1 = F^\mu(\emptyset)$. Likewise, $E_{N+1,N} = \emptyset$.

Our general case of interest is where we only assume $\lambda_N^\mu \neq 0$. Bear in mind that in the sequence $C(I_{\ell,N})$, $c_0 = b_i$ signals that the corresponding μ_i, taking part in $F(I_{\ell,N})$, can be moved all the way to the right without changing $F(I_{\ell,N})$. If we allow $\lambda_i^\mu = 0$ this means that such elements, when applied to v_0, give zero. Hence if we let $\mathcal{Z}_{\ell,N} = \{i = \ell, \cdots, N \mid \lambda_i^\mu = 0\}$ and if we let $\mathcal{J}_{\ell,N}^{\mathcal{Z}}$ denote those sequences I in $\mathcal{J}_{\ell,N}$ for which any index i from $\mathcal{Z}_{\ell,N}$ either belongs to an increasing sequence or is the biggest index in a decreasing sequence, then we have:

Proposition 6.4.

$$\{F^\mu(I_{\ell,N}) v_0 \mid I_{\ell,N} \in \mathcal{J}_{\ell,N}^{\mathcal{Z}}\} \tag{42}$$

is a basis of $V^{\Lambda_L - \mu_\ell + \cdots - \mu_N}$.

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Clearly there is an analogue to Proposition 6.3 for this general case (just as long as $\lambda^\mu_N > 0$).

There is yet another helpful way to view the various sets $\mathcal{J}_{\ell,N}$, $\ell = N, N-1, \dots, 1$, namely as a labeled, directed rooted tree with root at F_{μ_N}:

$$\begin{tikzcd} & F^\mu(I_{\ell,N}) \arrow[dl, "L_{\ell-1}"'] \arrow[dr, "R_{\ell-1}"] & \\ F_{\mu_{\ell-1}}F^\mu(I_{\ell,N}) & & F^\mu(I_{\ell,N})F_{\mu_{\ell-1}} \end{tikzcd} . \tag{43}$$

Here, it is really only the relative positions of F_{μ_ℓ} and $F_{\mu_{\ell-1}}$ that matter.

If we have $\lambda^\mu_i = 0$ we just modify the tree by removing all branches labeled by R_i - as well as everything above these branches - from the tree. (In this picture, the root is lowest.)

In this way, there is an obvious bijection between the paths in the modified tree and the basis.

We now return to (37). To obtain the following equations, it is used that $E_{\mu_{i-1}}(W_{i,j}) = -qW_{i-1,j}K_{\mu_{i-1}} + (W_{i,j})E_{\mu_{i-1}}$, which follows from Lemma 2.1. Furthermore, for the vector in (37) to be a $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ highest weight vector we clearly only need to look at $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}}_L)$. Here we must have:

$$\begin{aligned} \forall i = 1, \dots, N : (-q)W_{i,1}K_{\mu_i}u_{i+1}v_0 + W_{i,1}E_{\mu_i}u_iv_0 &= 0 & (44)\\ \forall i,j = 1,\dots,N : E_{\mu_j}u_iv_0 &= 0 \text{ if } i \neq j. & (45) \end{aligned}$$

We assume throughout that $\lambda^\mu_N \neq 0$.

Using Proposition 6.3, we set $u_{N+1} = 1$ and

$$\forall i = 1, \dots, N, u_i := a_iF_{\mu_i}u_{i+1} + b_iu_{i+1}F_{\mu_i} \text{ (except } b_N := 0). \tag{46}$$

Lemma 6.5. *The vector v_1 in (37) is a highest weight vector if and only if*

$$a_N = \frac{q^{1+\lambda^\mu_N}}{[\lambda^\mu_N]}, \tag{47}$$

$$(a_{N-1}[\lambda^\mu_{N-1}+1] + b_{N-1}[\lambda^\mu_{N-1}])u_Nv_0 = q^{\lambda^\mu_{N-1}+2}u_Nv_0, \tag{48}$$

$$(a_{N-1}[\lambda^\mu_N] + b_{N-1}[\lambda^\mu_N+1])F_{\mu_{N-1}}u_{N+1}v_0 = 0. \tag{49}$$

$$\textit{For } i < N-1: \tag{50}$$

$$(a_i[\lambda^\mu_i+1] + b_i[\lambda^\mu_i])u_{i+1}v_0 = q^{\lambda^\mu_i+2}u_{i+1}v_0, \tag{51}$$

$$\begin{aligned} &\left(a_i(a_{i+1}[\lambda^\mu_{i+1}+1] + b_{i+1}[\lambda^\mu_{i+1}])\right)F_{\mu_i}u_{i+2}v_0 + & (52)\\ &\left(b_i(a_{i+1}[\lambda^\mu_{i+1}+2] + b_{i+1}[\lambda^\mu_{i+1}+1])\right)F_{\mu_i}u_{i+2}v_0 = 0. \end{aligned}$$

In continuation of the discussion following Proposition 6.4, notice that if $\lambda^\mu_i = 0$ then equation (53) should be stricken, $b_i = 0$, and $a_i = q^2$.

Returning to the general case: If all $\lambda^\mu_i \neq 0$:

$$\begin{aligned} a_N &= \frac{q^{\lambda^\mu_N+1}}{[\lambda^\mu_N]}. & (53)\\ \forall k = 1,\dots,N-1 : && (54)\\ a_{N-k} &= q^{\lambda^\mu_{N-k}+2}\frac{[\lambda^\mu_N + \cdots + \lambda^\mu_{N-k+1} + k]}{[\lambda^\mu_N + \cdots + \lambda^\mu_{N-k+1} + \lambda^\mu_{N-k} + k]}, & (55)\\ b_{N-k} &= -q^{\lambda^\mu_{N-k}+2}\frac{[\lambda^\mu_N + \lambda^\mu_{N-k+1} + k - 1]}{[\lambda^\mu_N + \cdots + \lambda^\mu_{N-k+1} + \lambda^\mu_{N-k} + k]}. & (56) \end{aligned}$$

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012055 doi:10.1088/1742-6596/1194/1/012055

If $\lambda^\mu_{N-1} = 0 = \cdots = \lambda^\mu_{N-R}$ the a_{N-k} just become q^2 for $k = 1, \ldots, R$. This is just the limit of the equations (55). The corresponding $b_{N-k} = 0$ seemingly do not have a nice limit, but recall that instead, we just cut all branches of the tree marked by R_{N-i}, $i = 1 \ldots, R$. Actually, in this sense there is a nice limit for any case in which $\lambda^\mu_i = 0$ for some values of $i = 1, \ldots, N-1$.

7. One-sidedness

Recall that $\det_q$ is central in $\mathcal{A}^-$.

Proposition 7.1 (One-sidedness)**.** *If* $\Lambda = (\Lambda_L, \Lambda_R, \lambda)$ *and if*

$$\phi^1_{\Lambda_{n-1},\Lambda_n} \circ \phi^1_{\Lambda_{n-2},\Lambda_{n-1}} \circ \cdots \circ \phi^1_{\Lambda,\Lambda_1} = \mathrm{Det}_q, \tag{57}$$

where Det_q *denotes the operator* $M(V_\Lambda) \ni p \to \det_q \cdot p \in M(V_{\Lambda_n})$, *then at least one of the pair* Λ_L, Λ_R *is 0.*

We call such a representation *one-sided.* We shall see later that there is a converse to this.

Proof. The proof (sketched) is obtained in 10 installments:
1. We shall need the following elementary result:

Lemma 7.2. *Let* $a, b \in \mathbb{N}$ *with* $b \leq a$. *Then*

$$[a]_q[b]_q = [a+b-1]_q + [a+b-3]_q + \cdots + [a-b+1]_q.$$

Proof of Lemma: Using that $[a+1]_q = q^{-a} + q^{-a+2} + \cdots + q^a$, this follows easily by counting q exponents. □

2. We have that $\det_q \otimes V \subseteq \mathcal{A}^-_1 \otimes \mathcal{A}^-_{n-1} \otimes V = \mathcal{A}^-_1 \otimes (\mathcal{A}^-_{n-1} \otimes V) = (\mathcal{A}^-_1 \otimes \mathcal{A}^-_{n-1}) \otimes V$ and $\mathcal{A}^-_{n-1}$ is a sum of double tableaux of box size $(n-1) \times (n-1)$ and similarly $\mathcal{A}^-_n$ is a sum of double tableaux of box size $(n) \times (n)$. By the Littlewood-Richardson rule, to get $\det_q$ we need to use the invariant subspace $\mathcal{A}^-_{n-1}(n-1)$ of $(n-1) \times (n-1)$ minors in $\mathcal{A}^-_{n-1} \otimes V$. We can ignore contributions from other minors.
3. We now extend the notation used in Proposition 6.3 to also cover the cases of representations of $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}}_R)$ in the obvious way. We then have the following extension of said proposition:

$$\text{If} \quad v_1 = \sum_{k=1,\ell=1}^{i+1,j+1} W_{k,\ell} u_{k,\ell} \tag{58}$$

is a highest weight vector and $u_{i+1,j+1} = 1$, then

$$\forall k = 1, \ldots, i+1, \ell = 1, \ldots, j+1 : \tag{59}$$
$$u_{k\ell} = \sum_{I_{k,i} \in \mathcal{J}_{k,i}, I_{\ell,j} \in \mathcal{J}_{\ell,j}} f^\mu(C^\mu(I_{k,i})) f^\nu(C^\nu(I_{\ell,j})) F^\mu(I_{k,i}) F^\nu(I_{\ell,j}) v_0.$$

4. Let $\mathcal{A}^-_{n-1}(n-1)$ denote the space generated by the $(n-1) \times (n-1)$ minors in $\mathcal{A}^-$. This is a $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ module of highest weight $\Lambda^\mu = (0, 0, \ldots, 0, 1) = \Lambda^\nu$. The same kind of reasoning can be applied to $\mathcal{A}^-_{n-1}(n-1) \otimes V$. (Notice that $\mathcal{A}^-_{n-1}(n-1)$ is the dual to $\mathcal{A}^-_1$.) A highest weight vector v_0 in an irreducible submodule $V_0 \subseteq \mathcal{A}^-_{n-1}(n-1) \otimes \tilde{V}$ has the form

$$v_0 = \sum_{k,\ell} A(a+k, b+\ell) \tilde{u}_{a+k,b+\ell} \tilde{v}_0, \tag{60}$$

where the vectors $\tilde{u}_{a+k,b+\ell}\tilde{v}_0$, if $k+\ell>0$, have weights strictly smaller that $\tilde{v}_0$.
5. If we insert (60) into (58) and isolate the $\tilde{v}_0$ terms, we get in particular, using (57, (59), and since clearly here $(a,b)=(i+1,j+1)$ that

$$\sum_{k=1}^{i+1}\sum_{\ell=1}^{j+1} W_{k,\ell}\sum_{I_{k,i}\in\mathcal{J}_{k,i},I_{\ell,j}\in\mathcal{J}_{\ell,j}} f^\mu(C^\mu(I_{k,i}))f^\nu(C^\nu(I_{\ell,j}))F^\mu(I_{k,i})F^\nu(I_{\ell,j})A(i+1,j+1)=\kappa\cdot\det{}_q \quad (61)$$

for some constant $\kappa\neq 0$. It is easy to see that $F^\mu(I_{k,i})F^\nu(I_{\ell,j})A(i+1,j+1))=0$ unless $(I_{k,i}),I_{\ell,j})=(E_{k,i},E_{\ell,j})$. In the latter case we get, by (74) in Chapter 1, $(-q^{-1})^{i+j-k-\ell}A(k,\ell)$.
So

$$\sum_{k=1}^{i+1}\sum_{\ell=1}^{j+1} W_{k,\ell}(-q^{-1})^{i+j-k-\ell}f^\mu(C^\mu(E_{k,i}))f^\nu(C^\nu(E_{\ell,j})A(k,\ell)=\kappa\cdot\det{}_q. \quad (62)$$

6. If both $i+1<n$ and $j-1<n$ we can apply $F_{\nu_{n-1}}\dots F_{\nu_{j+1}}F_{\mu_{n-1}}\dots F_{\mu_{i+1}}$ to both sides of (62) and get that $W_{n,n}A(i+1,j+1)=0$; a contradiction.
7. Let us first assume that $i=j=n$. If we set $d_{k,\ell}=f^\mu(C^\mu(E_{k,i}))f^\nu(C^\nu(E_{\ell,j})$, (62) becomes

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n} W_{k,\ell}d_{k,\ell}(-q^{-1})^{2n-k-\ell}A(k,\ell)=\kappa\cdot\det{}_q. \quad (63)$$

Using (35) we can subtract a certain multiple of $\det_q$ in each row such that in the resulting equations

$$\sum_{k=0}^{n-1}\sum_{\ell=0}^{n-1} W_{n-k,n-\ell}b_{k,\ell}A(n-k,n-\ell)=\tilde{\kappa}\cdot\det{}_q, \quad (64)$$

we may assume: $\forall k: b_{k,n}=0$. Of course, this may change the constant into $\tilde{\kappa}$. A) If all the remaining $b_{k,\ell}$s are zero then, naturally, the resulting $\tilde{\kappa}$ is zero but that will also imply that each row of the original system satisfies, up to a constant non-zero multiple, equation (35). In particular,

$$\sum_{k=1}^{n} W_{k,n}d_{k,n}(-q^{-1})^{n-k}A(k,n)=\tilde{\kappa}\cdot\det{}_q. \quad (65)$$

B) If a non-zero system remains, we can subtract using column equations (36) to remove the terms $W_{nj}A(n,j)$; $j=1,\dots,n-1$ (the term with $j=n$ has already been removed. If there still remains an equation

$$\sum_{k=0}^{i-1}\sum_{\ell=0}^{j-1} W_{i-k,j-\ell}b_{k,\ell}A(i-k,j-\ell)=\kappa\cdot\det{}_q, \quad (66)$$

we reach a contradiction as in **6**.
In conclusion:
There is either a column equation

$$\sum_{k=1}^{n} W_{k,n}d_{k,n}(-q^{-1})^{n-k}A(k,n)=\tilde{\kappa}\cdot\det{}_q, \quad (67)$$

or an analogous row equation

$$\sum_{\ell=1}^{n} W_{n,\ell}d_{n,\ell}(-q^{-1})^{n-\ell}A(n,\ell)=\tilde{\kappa}\cdot\det{}_q. \quad (68)$$

8. Suppose that we have a row equation

Lemma 7.3.

$$\sum_{\ell=1}^{n} W_{n,\ell} d_{n,\ell}(-q^{-1})^{n-\ell} A(n,\ell) = \tilde{\kappa} \cdot \det_q. \tag{69}$$

Then $\lambda^{\mu}_{n-1} = 1$ *and* $\forall i = 1, \dots, n-2 : \lambda^{\mu}_i = 0$.

Proof. We have a PBW basis made up of monomials $W_{n,j_n,W_{n,j_{n-1}}}, \dots, W_{1,j_1}$. It follows that $\kappa = 1$ and it follows from (69) and (35) that $\forall k = d_{k,n} = q^{2(n-k)}$. It is easy to see (see **7.**) that $d_{k,n} = a_{n-1}a_{n-2}\cdots a_k$. This clearly implies that $a_k = q^2$ for all $k = 1, \dots .n-1$.

In particular, $a_{n-1} = q^2$, hence

$$q^2 = \frac{q^{1+\lambda^{\mu}_{n-1}}}{[\lambda^{\mu}_{n-1}]_q} \Rightarrow q^{2\lambda^{\mu}_{n-1}-2} = 1 \Rightarrow \lambda^{\mu}_{n-1} = 1.$$

Inductively, it follows from (55) that

$$q^{\lambda^{\mu}_{N-k}} \frac{[1+k]}{[\lambda^{\mu}_{N-k}+1+k]} = 1 \Rightarrow \lambda^{\mu}_{N-k} = 0. \tag{70}$$

9. If there is a column equation, it follows in the same way that $\Lambda_R = (0, 0, \dots, 0, 1)$.

10. By **6, 7** what remains are the cases $i < n, j = n$ and $i = n, j < n$. However, it is clear that they, by inspection, are covered by the arguments of the case $i = j = n$ simply by eliminating one possibility, so that if $j = n$, we must have $\Lambda_R = 0$ and if $i = n$ we must have $\lambda_L = 0$. □

We have the following converse which is quite straightforward:

Proposition 7.4. *Let* $V_\Lambda = V(\Lambda_L, 0, \lambda)$. *Set* $\Lambda_0 = \Lambda$. *Then there exist* $\mathcal{A}_q^-\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ *intertwining maps maps* $\psi^1_{\Lambda_i,\Lambda_{i+1}} : V_{\Lambda_i} \to V_{\Lambda_{i+1}} \subset V_{\Lambda_i} \otimes \mathcal{A}_q^-(1)$, *for* $i = 0, 1 \dots, n-1$, *independent of* λ, *such that, with* $\Lambda_n = (\Lambda^0_\mu, 0, \lambda + 2)$,

$$\psi^1_{\Lambda_{n-1},\Lambda_n} \circ \psi^1_{\Lambda_{n-2},\Lambda_{n-1}} \circ \cdots \circ \psi^1_{\Lambda,\Lambda_1} = \mathrm{Det}_q.$$

This decomposition is not unique. Furthermore the maps may be grouped together to form maps of higher degrees, defined by means of minors of the given degree.

8. First order intertwiners

It is clear that any submodule V_{Λ_1} of $\mathcal{A}_q^-(1) \otimes V_\Lambda$ defines a $\mathcal{A}_q^-\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ equivariant map $M(V_{\Lambda_1}) \to M(V_\Lambda)$. We shall now see that there is a unique $\lambda = \lambda(\Lambda_L, \Lambda_R)$ for which this becomes a $\mathcal{U}_q(\mathfrak{g}^{\mathbb{C}})$ equivariant map. See our forthcoming article for details. Notice also that the integrality assumption on (Λ_L, Λ_R) is not used.

We need the following extra information. Modulo $\mathcal{A}_q^- E_\beta$ it holds:

$$Z_\beta(W_{i,1}) = T_{\mu_{i-1}} T_{\mu_{i-2}} \dots T_{\mu_2}(F_{\mu_1}) K_\beta^{-1}, \tag{71}$$

$$Z_\beta(W_{i,j}) = -(q-q^{-1}) T_{\nu_{j-1}} T_{\nu_{j-2}} \dots T_{\nu_2}(F_{\nu_1}) T_{\mu_{i-1}} T_{\mu_{i-2}} \dots T_{\mu_2}(F_{\mu_1}) K_\beta^{-1} \text{ if } i,j \geq 2. \tag{72}$$

Proposition 8.1. *To any* $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ *homomorphism* $V_{\Lambda_1} \to \mathcal{A}_q^-(1) \otimes V_\Lambda$ *there corresponds a unique* λ *such that* $\psi_{\Lambda_1,\Lambda} \in Hom_{\mathcal{U}_q(\mathfrak{g}^{\mathbb{C}})}(M(V_{\Lambda_1}), M(V_\Lambda))$.

We focus on the case where $\Lambda = (\Lambda_L, 0, \lambda)$. Recall (29) and consider

$$Z_\beta(W_{N+1}v_0 + W_N u_N v_0 + W_{N-1}u_{N-1}v_0 + W_{N-2}u_{N-2}v_0 + \cdots + W_1 u_1 v_0) = \quad (73)$$

$$q^{-\lambda}\sum_{k=0}^{N-1} T_{\mu_{N-k}}\cdots T_{\mu_2}T_{\mu_1}(F_{\mu_1})u_{N-k+1}v_0 + [\lambda_1+1]u_1v_0 = 0. \quad (74)$$

We may expand the equation into equations for each vector $F^\mu(I_{1,N})$ in the basis. We claim that the general case can be reduced by contraction of trees to just the equation for $F^\mu(E_{1,N}) = F_{\mu_1}F_{\mu_2}\cdots F_{\mu_N}v_0$. Here we get

$$q^{-\lambda}(1-a_N+a_Na_{N-1}+a_Na_{N-1}a_{N-2}+\cdots+a_Na_{N-1}a_{N-2}\cdots a_2+[\lambda+1]a_Na_{N-1}a_{N-2}\cdots a_2a_1 = 0. \quad (75)$$

$$\begin{aligned}
1+a_N &= q\frac{[\lambda^\mu_N+1]}{[\lambda^\mu_N]}, & (76)\\
1+a_N+a_Na_{N-1} &= q^2\frac{[\lambda^\mu_N+1]}{[\lambda^\mu_N]}\frac{[\lambda^\mu_N+\lambda^\mu_{N-1}+2]}{[\lambda^\mu_N+\lambda^\mu_{N-1}+1]}, & (77)\\
1+a_N+a_Na_{N-1}+a_Na_{N-1}a_{N-2} &= q^3\frac{[\lambda^\mu_N+1]}{[\lambda^\mu_N]}\frac{[\lambda^\mu_N+\lambda^\mu_{N-1}+2]}{[\lambda^\mu_N+\lambda^\mu_{N-1}+1]}\frac{[\lambda^\mu_N+\lambda^\mu_{N-1}+\lambda^\mu_{N-2}+3]}{[\lambda^\mu_N+\lambda^\mu_{N-1}+\lambda^\mu_{N-2}+2]}. & (78)
\end{aligned}$$

$$S := 1+a_N+a_Na_{N-1}+a_Na_{N-1}a_{N-2}+\cdots+a_Na_{N-1}a_{N-2}\cdots a_2 = \quad (79)$$

$$q^{N-1}\frac{[\lambda^\mu_N+1]}{[\lambda^\mu_N]}\cdots\frac{[\lambda^\mu_N+\cdots+\lambda^\mu_{N-k}+k+1]}{[\lambda^\mu_N+\cdots+\lambda^\mu_{N-k}+k]}\cdots\frac{[\lambda^\mu_N+\lambda^\mu_{N-1}+\cdots+\lambda^\mu_2+N-1]}{[\lambda^\mu_N+\lambda^\mu_{N-1}+\cdots+\lambda^\mu_2+N-2]}. \quad (80)$$

Comparing to

$$T := a_Na_{N-1}a_{N-2}\cdots a_2a_1, \quad (81)$$

one easily obtains

$$q^{-\lambda}S + [\lambda+1]T = 0, \quad (82)$$

which upon divison becomes

$$q^{-\lambda} + [\lambda+1]q^{\lambda^\mu_1+\lambda^\mu_2+\cdots+\lambda^\mu_N+N}\frac{1}{[\lambda^\mu_1+\lambda^\mu_2+\cdots+\lambda^\mu_N+N-1]} = 0. \quad (83)$$

Using the equation $[a+b] = q^{-a}[b] + q^b[a]$, one easily concludes:

$$[\lambda+\lambda^\mu_1+\lambda^\mu_2+\cdots+\lambda^\mu_N+N] = 0. \quad (84)$$

This result can easily be generalized to the general first order case. It is related to the q-Shapovalov form [5].

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